Floer theory course notes

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Lecture I

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Disclaimer: these notes are an almost verbatim AI transcription of the author's handwritten notes, written on an iPad. Apart from formatting and the correction of a few small typos, the content is unchanged.

1 Definition and examples

We start with the following.

Definition 1.1. A *symplectic manifold* is a smooth manifold X together with a *symplectic form* ω, i.e. a 2-form such that:

(i) ω is closed, $d\omega = 0$;

(ii) ω is non-degenerate, i.e. the musical map

$$\omega^\sharp \colon TX \longrightarrow T^*X, \qquad v \longmapsto \omega(v, -)$$

is an isomorphism.

The first example we will look at is that of cotangent bundles. These correspond to phase spaces in mechanics and motivate many of the notions in symplectic geometry.

Given M a closed manifold, T^*M consists, as a set, of pairs (q, p) where

$$q \in M, \qquad p \colon T_qM \to \mathbb{R} \text{ linear.}$$

There is a so-called *canonical* or *Liouville* 1-form λ given by

$$\lambda_{(q,p)}(\xi) \coloneqq p\big(\pi_*(\xi)\big),$$

where $\pi \colon T^*M \to M$ is the projection. In local coordinates,

$$\lambda = \sum_{i=1}^{\dim M} p_i \, dq_i.$$

Proposition 1.2. *The form $d\lambda = \sum_i dp_i \wedge dq_i$ is symplectic.*

Proof. Since $d^2 = 0$, we have $d(d\lambda) = 0$, so $d\lambda$ is closed.

To show it is non-degenerate, let

$$v = \sum_i \Big(\alpha_i \frac{\partial}{\partial q_i} + \beta_i \frac{\partial}{\partial p_i} \Big)$$

be a tangent vector that is non-zero. Assume $\alpha_j \neq 0$ and let $w = \frac{\partial}{\partial p_j}$. Then

$$\begin{aligned} d\lambda(v,w) &= \sum_{i=1}^{\dim M} dp_i \wedge dq_i\,(v,w) \\ &= dp_j \wedge dq_j\Big(\alpha_j \frac{\partial}{\partial q_j} + \beta_j \frac{\partial}{\partial p_j},\, \frac{\partial}{\partial p_j}\Big) \\ &= -\alpha_j \neq 0. \end{aligned}$$

(If instead all α_i vanish, then some $\beta_j \neq 0$, and $w = \frac{\partial}{\partial q_j}$ gives $d\lambda(v,w) = \beta_j \neq 0$.) □

Another example is $\mathbb{C}^n \cong T^*\mathbb{R}^n$. But also any smooth projective variety (since $\mathbb{CP}^n$ is symplectic).

Remark 1.3. Being symplectic implies $\dim X = 2n$ for some n.

2 Special submanifolds and Hamiltonian isotopies

Let $C \subset X$ be an embedded submanifold of X. Define the sub-bundle $(TC)^\omega \subset TX|_C$ given by

$$\big\{\, v \in TX|_C \;\big|\; \omega(v,w) = 0 \quad \forall\, w \in TC \,\big\},$$

the *symplectic orthogonal* of C.

Definition 2.1. C is said to be:

(i) *Isotropic* if $TC \subset (TC)^\omega$; this is equivalent to $\omega|_{TC} = 0$. In this case $\dim C \leq \frac{1}{2}\dim X$.

(ii) *Coisotropic* if $(TC)^\omega \subset TC$; in this case $\dim C \geq \frac{1}{2}\dim X$.

(iii) *Lagrangian* if it is both, i.e. $\dim C = \frac{1}{2}\dim X$ and $\omega|_{TC} = 0$.

Examples.

(i) Any $N \subset M \subset T^*M$, seen as a submanifold of the zero-section, is isotropic.

(ii) $S^{2n-1} \subset \mathbb{C}^n$ is coisotropic.

(iii) $0_M \subset T^*M$ is Lagrangian.
$\mathbb{RP}^n \subset \mathbb{CP}^n$ is Lagrangian.

Remark 2.2. If C is coisotropic, the distribution $(TC)^\omega$ is integrable (Frobenius) to a foliation $\mathcal{C}$, called the *characteristic foliation*. If $C/\mathcal{C}$ is smooth, then it is automatically symplectic.

Let $H_t \colon X \to \mathbb{R}$, $t \in [0,1]$, be smooth. Since ω is non-degenerate, one can solve

$$\omega(X_{H_t}, -) = dH_t(-)$$

uniquely. This is the *Hamiltonian vector field* associated to H_t, and H_t is the *Hamiltonian* associated to X_{H_t}.

Example 2.3. In $\mathbb{R}^{2n}$, the system

$$\omega(X_H, -) = dH(-)$$

amounts to the usual Hamilton equations.

The flow φ^t of X_{H_t} is called a *Hamiltonian isotopy* and φ^1 is called a *Hamiltonian diffeomorphism* (time 1 is only a convention). In particular, φ^1 is a *symplectomorphism*, that is, $(\varphi^1)^*\omega = \omega$.

3 Main motivating example

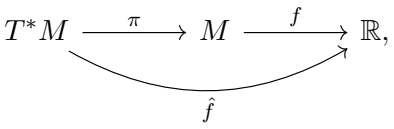
Recall that in T^*M, $\omega = d\lambda \rightsquigarrow$ *exact symplectic manifold*.

Definition 3.1. A Lagrangian $L \subset T^*M$ is *exact* if $\lambda|_L = df$ for some $f\colon L \to \mathbb{R}$.

For example, if $f\colon M \to \mathbb{R}$ is smooth, then

$$\Gamma(df) = \{\, (q, df_q) \mid q \in M \,\}$$

is exact. Moreover, if one sets $\hat{f} \coloneqq f \circ \pi$,



then the associated φ^1 satisfies

$$\varphi^1(0_M) = \Gamma(df).$$

That is, $\Gamma(df)$ is *Hamiltonian isotopic* to the zero section.

Observe:

$$\Gamma(df) \cap 0_M \cong \operatorname{crit}(f),$$

and $\Gamma(df) \pitchfork 0_M$ if and only if f is *Morse*.

But we know:

Theorem 3.2 (Morse inequalities)**.** *If $f\colon M \to \mathbb{R}$ is Morse, then*

$$\#\operatorname{crit}(f) \;\geq\; \sum_{i=0}^{\dim M} \dim H^i(M; \mathbb{Z}_2).$$

This motivated:

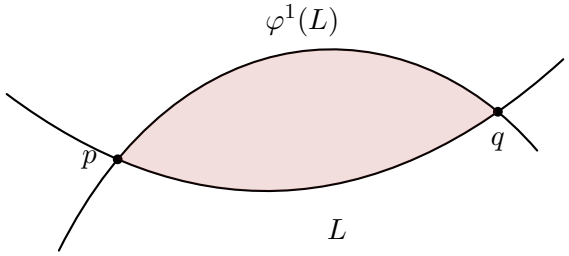
Conjecture 3.3 (Arnol'd)**.** *If L is a closed exact Lagrangian in an exact symplectic manifold*[1]*, then for every Hamiltonian diffeomorphism φ such that $\varphi(L) \pitchfork L$,*

$$\#\big(\varphi(L) \cap L\big) \;\geq\; \sum_i \dim H^i(L; \mathbb{Z}_2).$$

This conjecture takes many forms; in this instance it has been solved.

Theorem 3.4 (Floer)**.** *This is true.*

The idea behind it is, as in Morse theory, to build a chain complex spanned by $L \cap \varphi^1(L)$ and whose differential *counts* certain "things": in our case, *Floer strips*:



We will make this precise in the next lecture.

[1] We actually need the symplectic manifold to be *convex at infinity*; we will explain this later in the course.