Floer theory course notes

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Lecture II: Morse theory recap

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Disclaimer: these notes are an almost verbatim AI transcription of the author's handwritten notes, written on an iPad. Apart from formatting and the correction of a few small typos, the content is unchanged.

In the previous lecture we saw that Morse theory, in particular Morse (co)homology, inspired the Arnol'd conjecture and Floer theory.

It turns out that, while Floer theory is much harder and has its own set of new challenges, the *underlying principles* are roughly the same. That is why Floer theory is sometimes called *infinite dimensional Morse theory* (see for example Floer's 1989 paper "Witten's complex and infinite-dimensional Morse theory" [2]).

This is why in this lecture we recap the basics of Morse homology.

1 Morse functions

Let M be a closed, smooth manifold and $f\colon M \to \mathbb{R}$ a smooth function.

If $p \in \mathrm{crit}(f)$, the *Hessian* of f at p is well-defined and symmetric: for $v, w \in T_pM$ set

$$\begin{aligned}\mathrm{Hess}_p(f)(v,w) &= \tilde{v}_p\big(\tilde{w}\cdot f\big)\\ &= d_p\big(df(\tilde{w})\big)(v),\end{aligned}$$

where $\tilde{v}, \tilde{w}$ are arbitrary extensions of v and w. It does not depend on $\tilde{v}$, since it is a differential at a point. It is symmetric, since

$$\tilde{v}_p(\tilde{w}\cdot f) - \tilde{w}_p(\tilde{v}\cdot f) = \underbrace{[\tilde{v},\tilde{w}]_p \cdot f}_{=0 \text{ since } d_pf=0}.$$

Hence it also does not depend on $\tilde{w}$.

Definition 1.1. The critical point p is *non-degenerate* if $\mathrm{Hess}_p(f)$ is non-degenerate. The function f is *Morse* if all its critical points are non-degenerate.

Theorem 1.2. *Morse functions are C^∞-generic: the space of Morse functions is open and dense.* ("Almost all functions are Morse".)

From now on, f is Morse. Given $p \in \mathrm{crit}(f)$, let $E^u(p)$ be a maximal subspace of T_pM on which $\mathrm{Hess}_p(f)$ is negative definite. The *Morse index* of f at p is then

$$\mathrm{ind}_f(p) := \dim E^u(p),$$

which does not depend on the choice of $E^u(p)$ (Sylvester's law of inertia).

Example 1.3. In $\mathbb{R}^n$ consider $f\colon \mathbb{R}^n \to \mathbb{R}$,

$$f(x_1, \dots, x_n) = -\sum_{i=1}^{k} x_i^2 + \sum_{j=k+1}^{n} x_j^2.$$

Then f is Morse: the only critical point is 0, and $\operatorname{ind}_f(0) = k$.

Actually, all Morse functions are *locally* like this.

Lemma 1.4 (Morse)**.** *If $p \in \operatorname{crit}(f)$ is non-degenerate, there exists a local coordinate system $(x_1, \dots, x_n)$ centred at p such that*

$$f(x) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{j=k+1}^{n} x_j^2,$$

where $k = \operatorname{ind}_f(p)$.

2 Gradient flow

Let g be a Riemannian metric. We can consider the *negative gradient flow*, given by solutions to the ODE

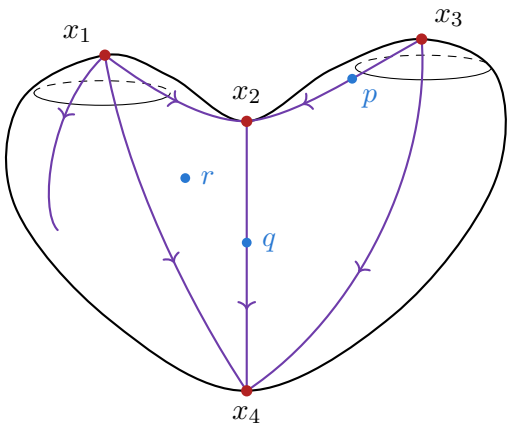
$$\dot{x} = -\nabla_g f.$$

Denote it by $\varphi^t(x)$. Given $p \in \operatorname{crit}(f)$, define:

$$W^u(p) = \left\{ x \in M \;\middle|\; \lim_{t \to -\infty} \varphi^t(x) = p \right\}, \qquad \text{the } \textit{unstable manifold} \text{ of } p,$$
$$W^s(p) = \left\{ x \in M \;\middle|\; \lim_{t \to +\infty} \varphi^t(x) = p \right\}, \qquad \text{the } \textit{stable manifold} \text{ of } p.$$

Elements of $W^u(p)$ come from p, while elements of $W^s(p)$ go to p.

Example 2.1. Consider the height function on the following deformed sphere, with two maxima x_1, x_3, a saddle x_2 and a minimum x_4:



Here

$$p \in W^u(x_3) \quad \text{and} \quad p \in W^s(x_2),$$
$$q \in W^u(x_2) \quad \text{and} \quad q \in W^s(x_4),$$
$$r \in W^u(x_1) \quad \text{and} \quad r \in W^s(x_4).$$

Every point in $M \setminus \operatorname{crit}(f)$ is in $W^u(p) \cap W^s(q)$ for some $p, q \in \operatorname{crit}(f)$.

3 Complex and differential

The *Morse complex* of f is easy to define:

$$MC_*(f) = \bigoplus_{p \in \mathrm{crit}(f)} \mathbb{Z}_2 \cdot p.$$

It is naturally graded by the Morse index: $MC_k(f) = \bigoplus_{\mathrm{ind}(p)=k} \mathbb{Z}_2 \cdot p$.

To define a differential $\partial\colon MC_*(f) \to MC_{*-1}(f)$ we want to "count trajectories from p to q". But for this we need this to be a finite set[1], and not only "by accident", but rather because it is *expected*.

4 Transversality

Definition 4.1. A pair (f, g) is *Morse–Smale* if for all $p, q \in \mathrm{crit}(f)$ the intersection $W^u(p) \cap W^s(q)$ is *transverse*.

This is a *robust* notion: if (f, g) is Morse–Smale, any sufficiently close pair will also be.

Theorem 4.2. *Given f, for* generic *g the pair (f, g) is Morse–Smale.*

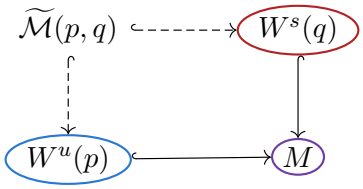
This is a *transversality statement*: by choosing g wisely, the *moduli spaces*

$$\widetilde{\mathcal{M}}(p, q) \coloneqq W^u(p) \cap W^s(q)$$

will be smooth manifolds (remember that a transverse intersection of submanifolds is a submanifold).

We can, and will, think of $\widetilde{\mathcal{M}}(p, q)$ also as the space of parametrised trajectories from p to q. Indeed, by existence and uniqueness of solutions to ODEs, to each $x \in \widetilde{\mathcal{M}}(p, q)$ we can associate $\gamma_x(t) = \varphi^t(x)$, and conversely $x = \gamma_x(0)$.

What is $\dim \widetilde{\mathcal{M}}(p, q)$? We have the intersection square



so that

$$\begin{aligned}\dim \widetilde{\mathcal{M}}(p, q) &= \dim \bigcirc + \dim \bigcirc - \dim \bigcirc \\ &= \dim W^s(q) + \dim W^u(p) - n \\ &= n - \mathrm{ind}(q) + \mathrm{ind}(p) - n \\ &= \mathrm{ind}(p) - \mathrm{ind}(q).\end{aligned}$$

Here we used that $\dim W^u(p) = \mathrm{ind}(p)$. This follows from the stable manifold theorem (see e.g. [4, Section 2.7]): the linearisation of $-\nabla_g f$ at p is symmetric with respect to g, so it diagonalises, and $W^u(p)$ is tangent at p to the sum of its eigenspaces with positive eigenvalues, which has dimension $\mathrm{ind}(p)$. The same argument shows that $\dim W^s(q) = n - \mathrm{ind}(q)$.

However, there is an annoying $\mathbb{R}$-action: if $\gamma\colon \mathbb{R} \to M$ is a solution in $\widetilde{\mathcal{M}}(p, q)$, then for all $r \in \mathbb{R}$

$$\tau_r \gamma(t) \coloneqq \gamma(t + r)$$

[1] As opposed to a positive-dimensional manifold.

is also a solution: we are just reparametrising. We want to say that all of these solutions are the same, so we set

$$\mathcal{M}(p,q) \coloneqq \widetilde{\mathcal{M}}(p,q)/\mathbb{R}.$$

Then

$$\dim \mathcal{M}(p,q) = \operatorname{ind}(p) - \operatorname{ind}(q) - 1.$$

In particular, if $\operatorname{ind}(p) = \operatorname{ind}(q) + 1$, then $\dim \mathcal{M}(p,q) = 0 \rightsquigarrow$ a 0-dimensional manifold, i.e. a discrete set.

We will show in the next section that it is finite. Meanwhile, we "define"

$$\partial\colon p \longmapsto \sum_{\substack{q \\ \operatorname{ind}(q)=\operatorname{ind}(p)-1}} \#_2\, \mathcal{M}(p,q) \cdot q.$$

5 Compactness

The spaces $\mathcal{M}(p,q)$ are usually *not compact*, but admit a natural compactification.

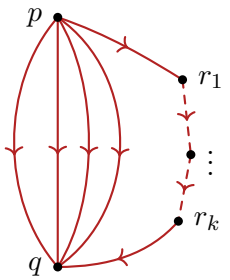
Definition 5.1. A *broken Morse trajectory* from p to q is a tuple

$$(\gamma_0, \dots, \gamma_k) \in \mathcal{M}(p, r_1) \times \mathcal{M}(r_1, r_2) \times \cdots \times \mathcal{M}(r_k, q), \qquad k \geq 1.$$

Theorem 5.2. *The spaces $\mathcal{M}(p,q)$ admit compactifications $\overline{\mathcal{M}}(p,q)$ with*

$$\operatorname{interior}\big(\overline{\mathcal{M}}(p,q)\big) = \mathcal{M}(p,q),$$
$$\partial\overline{\mathcal{M}}(p,q) = \{\textit{broken trajectories from } p \textit{ to } q\}.$$

This is a *compactness statement*:



Corollary 5.3. *If $\operatorname{ind}(q) = \operatorname{ind}(p) - 1$, then $\mathcal{M}(p,q)$ is compact, hence finite.*

Proof. Indeed, $\partial\overline{\mathcal{M}}(p,q)$ would consist of broken trajectories, but by index reasons there are none! □

Corollary 5.4. $\partial^2 = 0$.

Proof. Indeed,

$$\partial^2(p) = \sum_q \sum_r \#_2\mathcal{M}(p,r) \cdot \#_2\mathcal{M}(r,q) \cdot q.$$

Note that

$$\#_2\mathcal{M}(p,r) \cdot \#_2\mathcal{M}(r,q) = \#_2\big(\mathcal{M}(p,r) \times \mathcal{M}(r,q)\big).$$

Summing over r, this is the mod 2 count of broken trajectories from p to q. But this is $\partial\overline{\mathcal{M}}(p,q)$, and $\overline{\mathcal{M}}(p,q)$ is 1-dimensional. Any such (compact) manifold has an *even* number of boundary points! Hence

$$\#_2 \bigsqcup_r \mathcal{M}(p,r) \times \mathcal{M}(r,q) = 0.$$ □

Remark 5.5. There is an implicit *gluing* statement: every broken trajectory is a boundary point.

6 Conclusion

Define

$$MH_*(f) \coloneqq \frac{\ker \partial}{\operatorname{Im} \partial}.$$

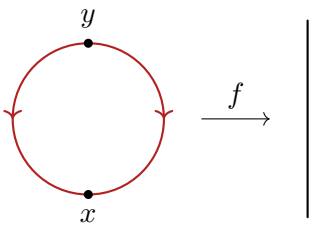
This is the *Morse homology* of f. Later we will see:

(i) It does not depend on f or g.

(ii) Actually, $MH_*(f) \cong H_*(M; \mathbb{Z}_2)$.

(iii) One can use other coefficients.

Example 6.1 (Final example)**.** Consider the height function on the circle:



Then

$$MC_*(f) = \underbrace{\mathbb{Z}_2 \cdot x}_{\text{index } 0} \oplus \underbrace{\mathbb{Z}_2 \cdot y}_{\text{index } 1}$$

and

$$\mathcal{M}(y,x) \;=\; \sqcup \quad \Longrightarrow \quad \#_2\mathcal{M}(y,x) = 1 + 1 = 0.$$

The complex is then

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \longrightarrow 0 \longrightarrow \cdots$$

so $MH_*(f) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong H_*(S^1; \mathbb{Z}_2)$.

References

[1] M. Audin and M. Damian, *Morse Theory and Floer Homology*, Universitext, Springer, 2014.

[2] A. Floer, *Witten's complex and infinite-dimensional Morse theory*, J. Differential Geom. **30** (1989), 207–221.

[3] J. Milnor, *Morse Theory*, Annals of Mathematics Studies 51, Princeton University Press, 1963.

[4] L. Perko, *Differential Equations and Dynamical Systems*, 3rd ed., Texts in Applied Mathematics 7, Springer, 2001.

[5] D. Salamon, *Morse theory, the Conley index and Floer homology*, Bull. London Math. Soc. **22** (1990), 113–140.