Floer theory course notes

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Lecture III: Finite-dimensional transversality

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Disclaimer: these notes are an almost verbatim AI transcription of the author's handwritten notes, written on an iPad. Apart from formatting and the correction of a few small typos, the content is unchanged.

In this lecture we will go into key concepts of Morse homology that carry over to Floer theory: the *transversality* results.

We will prove some important theorems of Morse theory in a definitely non-standard and painful route; however, the techniques generalise to Floer theory almost immediately.

1 First concepts

Let M, W and Z be closed manifolds, and $f\colon W \to M$, $g\colon Z \to M$ be smooth maps.

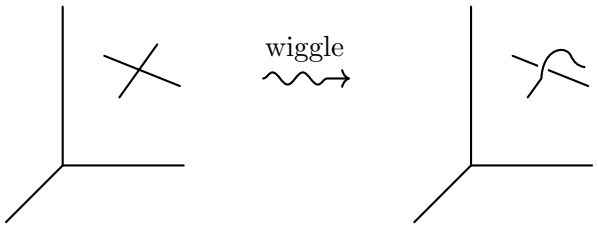
Definition 1.1. We say f and g are *transverse*, written $f \pitchfork g$, if for all $w \in W$, $z \in Z$ such that $f(w) = g(z) = x$ we have

$$\operatorname{Im}(d_w f) + \operatorname{Im}(d_z g) = T_x M. \tag{T}$$

Important observation. Suppose $\dim W + \dim Z < \dim M$. Then, by linear algebra, equation (T) is *never* satisfied... unless it is "vacuously true", that is:

Proposition 1.2. *If $\dim W + \dim Z < \dim M$, then $f \pitchfork g$ if and only if their images are disjoint.*

For example, two curves in $\mathbb{R}^3$ are transverse if and only if they don't intersect. We will see that this is partly because intersecting is not a robust/generic phenomenon: we can wiggle one of the curves so that they don't cross, and this wiggle can be as small as we want:



Arguably the most important theorem related to transverse maps is the following.

Theorem 1.3 (Implicit function)**.** *If $f \pitchfork g$, then the* fibered product

$$W \times_M Z \coloneqq \left\{ (w, z) \in W \times Z \;\middle|\; f(w) = g(z) \right\}$$

is naturally a smooth manifold, fitting in the square

$$\begin{array}{ccc} W \times_M Z & \xrightarrow{\mathrm{pr}_2} & Z \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle g} \\ W & \xrightarrow{f} & M \end{array}$$

Its dimension is

$$\dim W \times_M Z = \dim Z + \dim W - \dim M,$$

and the tangent space at (w, z) *is described by*

$$T_{(w,z)}\big(W \times_M Z\big) = \big\{\, (\xi, \eta) \in T_w W \times T_z Z \;\big|\; d_w f(\xi) = d_z g(\eta) \,\big\}.$$

There is an analogous statement for manifolds with boundary, and one with orientations (we will see this later).

As a particular case, recall that $f\colon M \to N$ is a *submersion* if for all $m \in M$, $d_m f$ is surjective. Then, for any (embedded) submanifold $Z \subset N$, the inclusion $\iota\colon Z \hookrightarrow N$ is transverse to f (property (T) is almost obvious). In particular,

$$M \times_N Z \cong f^{-1}(Z)$$

is a smooth manifold of dimension $\dim M + \dim Z - \dim N$.

For example, if $Z = \{n\} \subset N$ is a point, then the *fiber* $f^{-1}(n)$ is a smooth manifold of dimension $\dim M - \dim N$. Note that $T_m f^{-1}(n) \cong \ker d_m f$.

2 Sections and generic transversality

The most important framework we will study is based on the following toy model:

$$\begin{array}{c} E \\ \downarrow{\scriptstyle \pi} \\ M \end{array}$$

E is a vector bundle and $s\colon M \to E$ is a *section*, i.e. a smooth map such that $\pi \circ s = \mathrm{Id}_M$.

We always have the zero section $0_M\colon M \to E$, given by $0_M(m) = (m, 0)$.

If s is transverse to 0_M, then

$$M \times_E M \cong \big\{\, m \in M \;\big|\; s(m) = 0 \,\big\} =: s^{-1}(0)$$

is a smooth submanifold of M.

Example 2.1. Let M be closed and $E = TM$. Then a section is just a vector field V. V is transverse to the zero section if and only if it has finitely many, non-degenerate zeros.

Remark 2.2. Actually, in this case "$V^{-1}(0)$" is the signed[1] sum of zeros, and

$$[V^{-1}(0)] = \mathrm{PD}\big(\mathrm{Euler}(M)\big),$$

where Euler is the Euler class. This is behind the Poincaré–Hopf index theorem!

The cornerstone of differential topology is the following.

[1] There is an orientation to take care of.

Theorem 2.3 (Thom)**.** *Being transverse is a generic property, in the following sense: we may take arbitrarily small deformations of f, or g, or both, to make them transverse.*

Proof. See [1]. □

In the section setting we have the "variant":

Theorem 2.4. *Given any section s, we can find an arbitrarily close section s_ε so that s_ε is transverse to 0.*

There is another key theorem related to transversality. If $f\colon M \to N$ is smooth, then $n \in N$ is a *regular value* if for all $m \in f^{-1}(n)$, $d_m f$ is surjective. If this is not the case, n is a *critical value*.

Theorem 2.5 (Sard)**.** *The set of critical values has Lebesgue measure 0; in other words, it is "small".*

Proof. See [2]. □

The infinite-dimensional analogue is the *Sard–Smale* theorem, which is crucial for Floer theory.

3 An important situation

Let $E \to M$ be as before. Let $\mathcal{J}$ be another smooth manifold; we think of it as the "parameter space". Consider the vector bundle

$$\begin{array}{c} \pi_1^* E \\ \downarrow \\ M \times \mathcal{J} \end{array}$$

Given any section $s\colon M \times \mathcal{J} \to \pi_1^* E$ and any $j \in \mathcal{J}$, the map $s(-, j)\colon M \to E$ is a section. In some sense, we have a "family of sections".

Fix such an s. If s is transverse to $0_{M\times\mathcal{J}}$, then

$$s^{-1}(0) = \left\{ (m, j) \;\middle|\; s(m, j) = 0 \right\}$$

is a smooth manifold. Consider

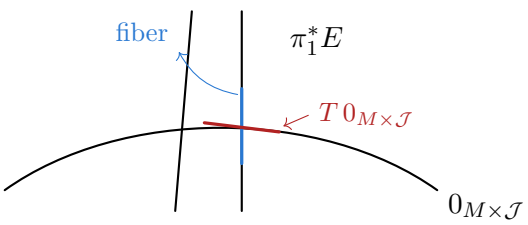
$$\pi_2|_{s^{-1}(0)}\colon s^{-1}(0) \longrightarrow \mathcal{J}, \qquad (m, j) \longmapsto j.$$

Proposition 3.1. *j is a regular value of $\pi_2|_{s^{-1}(0)}$ if and only if $s(-, j)$ is transverse to zero as a section of $E \to M$.*

Proof. Assume j is regular, and let m be such that $s_j(m) := s(m, j) = 0$.

At (m, j), we have a splitting

$$T_{(m,j)}\pi_1^* E \cong \left(\pi_1^* E\right)_{(m,j)} \oplus T_{(m,j)} 0_{M\times\mathcal{J}}.$$



The linear map ds also splits as

$$ds = ds_{\text{fiber}} \oplus ds_{\text{base}}.$$

It is easy to see that s is transverse to the zero section if and only if ds_{fiber} is surjective (at every zero of s).

Let $v \in \big(\pi_1^* E\big)_{(m,j)}$ be tangent to the fiber. Since s is transverse to zero, there exists $(\xi, \eta) \in T_{(m,j)}(M \times \mathcal{J})$ such that

$$ds_{\text{fiber}}(\xi, \eta) = v.$$

Since j is a regular value of $\pi_2|_{s^{-1}(0)}$, by definition there exists $(\xi', \eta') \in T_{(m,j)} s^{-1}(0)$, i.e. with $ds_{\text{fiber}}(\xi', \eta') = 0$, such that $\eta' = \eta$. Now

$$\begin{aligned} ds_{\text{fiber}}(\xi - \xi', 0) &= ds_{\text{fiber}}(\xi, \eta) - ds_{\text{fiber}}(\xi', \eta') \\ &= v - 0 = v. \end{aligned}$$

So $\xi - \xi'$ satisfies $d(s_j)_{\text{fiber}}(\xi - \xi') = v$, that is, $s_j \pitchfork 0_M$.

The converse is a backwards analogue. □

Corollary 3.2. *If s is transverse to $0_{M\times\mathcal{J}}$, then for almost all $j \in \mathcal{J}$, s_j is transverse to 0.*

Proof. Apply Sard's theorem. □

4 An example

Recall the following theorem.

Theorem 4.1. *The space of Morse functions is open and dense in $C^\infty(M)$ (with its C^∞ topology).*

We will give a proof of the "dense" part in the spirit of the previous situation.

Let $f\colon M \to \mathbb{R}$ be a smooth function. Consider the bundle $E = T^*M \to M$ and the section $df\colon M \to T^*M$. As a parameter space, pick some $\mathbb{R}^N$, where N is very large, to be determined.

Pick a point $p \in M$. In a neighbourhood U of p we can find $h_1, \dots, h_n\colon U \to \mathbb{R}$ such that $d_q h_1, \dots, d_q h_n$ are linearly independent for all $q \in U$. Indeed, one can take $h_i = x_i$ for some coordinates around p. Using bump functions (equal to 1 on a smaller neighbourhood V_p of p), extend this by 0 to all of M. The extended functions still have linearly independent differentials at every point of V_p.

Repeat this process to get a large, but finite, collection of functions $\{h_i\}_{i=1}^N$ so that for all $p \in M$ there exist $h_{i_1}, \dots, h_{i_n}$ such that $d_p h_{i_1}, \dots, d_p h_{i_n}$ are linearly independent. To see why we can find finitely many, invoke the compactness of M: finitely many of the V_p cover M.

Now consider the section

$$s\colon M \times \mathbb{R}^N \longrightarrow T^*M, \qquad (p, a_1, \dots, a_N) \longmapsto d_p f + d_p\Big(\sum_i a_i h_i\Big).$$

Claim. *s is transverse to 0.*

Indeed, pick $x = (p, a_1, \dots, a_N)$ so that $s(x) = 0$. By construction, there are $h_{i_1}, \dots, h_{i_n}$ so that $d_p h_{i_1}, \dots, d_p h_{i_n}$ are linearly independent. Then the vertical differential is surjective:

$$ds_{\text{fiber}}\colon \Big(\alpha_{i_1} \frac{\partial}{\partial a_{i_1}}, \dots, \alpha_{i_n} \frac{\partial}{\partial a_{i_n}}\Big) \longmapsto \alpha_{i_1}\, d_p h_{i_1} + \dots + \alpha_{i_n}\, d_p h_{i_n},$$

because the $d_p h_{i_j}$ form a basis of $T_p^* M$.

Then, applying the corollary, for almost all $(a_1, \dots, a_N) \in \mathbb{R}^N$ the section $d\big(f + \sum_i a_i h_i\big)$ is transverse to zero. In other words, $f + \sum_i a_i h_i$ is Morse.

The complement of a measure 0 set is dense. In particular, we can find arbitrarily small a_i so that $f + \sum_i a_i h_i$ is Morse. Since the h_i are fixed, $f + \sum_i a_i h_i \to f$ in the C^∞ topology as $a \to 0$. This shows the density.

For completeness, we show that the space of Morse functions is open.

If not, given f Morse, we could find a sequence of non-Morse functions $\{f_n\} \to f$. But then there exists a sequence of degenerate critical points $\{p_n\}$, $p_n \in \mathrm{crit}(f_n)$. By compactness of M, after passing to a subsequence, $\{p_n\} \to p$. But then $d_p f = 0$, so $p \in \mathrm{crit}(f)$.

Since the convergence is C^∞ (hence C^2), in a chart around p we have $\mathrm{Hess}_{p_n}(f_n) \to \mathrm{Hess}_p(f)$. But the latter is non-degenerate, and this is an open condition! Hence $\mathrm{Hess}_{p_n}(f_n)$ is non-degenerate for n large enough: a contradiction.

5 What's next?

Here we did everything in finite dimensions. In the next lecture we will work in the infinite-dimensional case, to show that the Morse–Smale condition is generic.

References

[1] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, 1974.

[2] J. Milnor, *Topology from the Differentiable Viewpoint*, University Press of Virginia, 1965.